JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 46, 143-168 (1974)

Generalized Projection Theorem and Weak Noncoercive Evolution Problems in Hilbert Space

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I

In this section we will first indicate the kind of problem to be solved and then discuss briefly how our results (partially reported in [24]) compare with some of the other known results in similar directions. Thus, let H be a separable Hilbert space over the complexes $\mathscr{C}$, and let

$$\{V(t) \mid t \in [0, T]\} \qquad (\text{with } 0 < T < \infty)$$

be a family of Hilbert spaces over $\mathscr{C}$, each $V(t)$ being a dense subspace of H with continuous-inclusion injection: $V(t) \to H$. In H the scalar product and norm are denoted by $(\cdot, \cdot)_H$ and $|\cdot|_H$; the corresponding quantities in $V(t)$ are respectively denoted by $((\cdot, \cdot))_t$ and $\|\cdot\|_t$. The symbol $|\cdot|$ will be reserved to indicate "absolute value." By $L^2(H)$ we will mean $L^2([0, T]; H)$, the space of equivalence classes of measurable square-integrable H-valued functions on $[0, T]$; following convention, such an equivalence class is denoted by an element belonging to it. Since H is separable, it suffices to call a function $f: [0, T] \to H$ measurable [4] if, $\forall h \in H$, the map

$$t \mapsto (f(t), h)_H: [0. T] \to \mathscr{C}$$

is (Lebesgue) measurable. It is well-known that $L^2(H)$ is a Hilbert space [21] under the scalar product $(\cdot, \cdot)_{L^2(H)}$ defined by

$$(f, g)_{L^2(H)} = \int_0^T (f(t), g(t))_H \, dt, \qquad \forall f, g \in L^2(H). \tag{1.1}$$

At this point it is convenient to introduce Carroll's standard operators $S(t): V(t) \to H$ developed and described in [4, 5, 6, 7, 8] (also cf. [14]). These operators, which exist under the assumptions above made on $V(t)$ and H, have a number of useful properties which we now mention: each $S(t)$ is a

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one-to-one, onto, positive, self-adjoint, linear, unbounded operator in H with domain $V(t)$. Further, the action of $S(t)$ is described by the relation,

$$((x, y))_t = (S(t)\,x, S(t)\,y)_H\,, \qquad \forall x, y \in V(t).$$

The use of standard operators $S(t)$ to describe and delimit the way in which $V(t)$ can vary in weak evolution problems was first introduced in [5, 6], and has since led to a number of existence, uniqueness, and regularity results in the variable domain situation under appropriate weak C^n (n = a nonnegative integer) hypotheses made on the family

$$\{S(t)^{-1} \mid t \in [0, T]\} \tag{1.2}$$

(cf. [4–6, 9–15, 24, 25, 28]). Related work in strong problems are found in [16, 17] and reported partially in [7]. Let us recall that by definition [4, 10, 24] the family (1.2) is measurable (respectively, weakly C^n) if for every pair of elements $h, k \in H$, the map

$$t \mapsto (S(t)^{-1}\,h, k)_H\colon [0, T] \to \mathscr{C}$$

is measurable (respectively, n times continuously differentiable) on $[0, T]$. We will always make assumptions so as to ensure that

the family (1.2) is uniformly bounded and measurable (on $[0, T]$) as operators in H. (1.3)

In particular, if the family (1.2) is weakly C^n for a nonnegative integer n (the case $n = 0$ having an obvious meaning), then the statement (1.3) is true (cf. [4, 24]). The following consequences of (1.3) will be used over and over again (cf. [4, 11, 24]):

$$W = S^{-1}(L^2(H)) = \{S(\cdot)^{-1} f(\cdot) \mid f(\cdot) \in L^2(H)\} \text{ is a subspace of } L^2(H), \tag{1.4}$$

and

W is a Hilbert space under the scalar product $((\cdot, \cdot))$ defined by

$$\begin{aligned} ((v, w)) &= (Sv, Sw)_{L^2(H)} \\ &= \int_0^T (S(t)\,v(t), S(t)\,w(t))_H\,dt \qquad \text{by (1.1).} \end{aligned} \tag{1.5}$$

References [4] and [8] give the relation of Carroll's standard operators to Schwarz's kernels.

The problem to be solved in this paper is a weak evolution problem of first order. It consists of showing the existence of a solution $u \in W$ (W is defined in (1.4) above) such that

$$
\begin{aligned}
- \int_0^T (u(t), v'(t))_H \, dt + \int_0^T a(t; u(t), v(t)) \, dt + \lambda \int_0^T (u(t), v(t))_H \, dt \\
= \int_0^T (f(t), v(t))_H \, dt + (u_0, v(0))_H, \qquad \forall v \in F,
\end{aligned}
\tag{1.6}
$$

where $f \in L^2(H)$ and $u_0 \in H$ are given satisfying appropriate regularity conditions, and

$$F = \{v \in W \mid v' \in L^2(H) \text{ and } v(T) = 0\}; \tag{1.7}$$

F will play a crucial role in our work. In (1.6) and (1.7) v' is the distribution derivative in $\mathscr{D}'((0, T); H)$, the space of H-valued distributions on $(0, T)$. Thus, point-values of $v \in W$ with $v' \in L^2(H)$ are defined by continuity (cf. [4]). Further, λ is a real number; and, $\forall t \in [0, T]$,

$$a(t; \cdot, \cdot): V(t) \times V(t) \to \mathscr{C}$$

is a continuous, sesquilinear form. If, $\forall t \in [0, T]$,

$$\mathscr{A}(t): V(t) \to V(t)$$

is the linear operator defined by

$$a(t; u, v) = ((\mathscr{A}(t)\, u, v))_t, \qquad \forall u, v \in V(t), \tag{1.8}$$

we will need to assume that

$$\text{the family } \{\mathscr{A}(t) \mid t \in [0, T]\} \text{ is measurable and uniformly bounded in operator norm (in } V(t)). \tag{1.9}$$

By the measurability condition we mean: if $t \mapsto \| u(t)\|_t: [0, T] \to \mathscr{R}$ is measurable, then $t \mapsto \| \mathscr{A}(t)\, u(t)\|_t: [0, T] \to \mathscr{R}$ is measurable, where $u: [0, T] \to H$ is a map with $u(t) \in V(t)$ a.e. in t. The conditions (1.9) thus imply that for every fixed pair $u, v \in W$, the map

$$t \mapsto a(t; u(t), v(t)): [0, T] \to \mathscr{C}$$

is measurable and bounded. We note in passing that (cf. [4, 11, 18, 24]) in (1.6) the presence of the term containing the fixed real number λ is merely a technical device; questions of existence or uniqueness remain unchanged if we put λ equal to zero or any other real number.

We will call the form $a(t; \cdot, \cdot)$ coercive if

$$\operatorname{Re} a(t; u, u) \geqslant \alpha \| u \|_t^2, \qquad \forall u \in V(t)$$

for some constant $\alpha > 0$. We will call the family $\{a(t; \cdot, \cdot) \mid t \in [0, T]\}$ coercive if α can be taken independent of $t \in [0, T]$. For such a coercive family the existence of a solution u of Eq. (1.6) is well-known, the existence being an immediate consequence of Lions' projection theorem (cf. [4, 18]). (Some uniqueness results in the variable domain situation are available in [9, 10, and 15].) It is now natural to pose the question whether Lions' projection theorem can be adapted or modified to yield existence results for Eq. (1.6) in cases of noncoercive forms $a(t; \cdot, \cdot)$. This question is answered in the affirmative in [11, 24] (cf. also [25]) wherein a systematic procedure is given by which, for certain specified types of noncoercivity, regularity of the data f and u_0 is exploited to reduce the problem of solving (1.6) to an equivalent problem expressible in terms of a new space of "test functions" Φ which takes the place of F (F is defined by (1.7)). One can then apply Lions' projection theorem to this equivalent problem. Such an approach is adopted in [12, 13, and 28] also.

The purpose of this paper is to provide, in Section III, an alternative approach to solving the existence problem for the equation (1.6). This alternative approach consists of applying a modified (in fact, a generalized) version of Lions' projection theorem to obtain existence of solutions directly. The types of noncoercivity considered as well as the nature of regularity conditions imposed on f and u_0 are different in scope from those appearing in [11], although some overlapping exists. Examples in Section III will make this clear. As we did in [11], we will provide examples to constant domain specializations of the more general variable domain results comprising Theorems 3.1 and 3.3. Thus, the methods of [11] and this paper are complementary to one another.

The generalized version of Lions' projection theorem mentioned in the preceding paragraph appears in detail in [27] (cf. also [24, 26]). However, for the sake of completeness, we will state and partly prove in Section II whatever we need in subsequent sections for application to evolution equations. The proof (of the generalized projection theorem) that we give is slightly different from and more general in scope than the proof appearing in [27]; the proof in this paper utilizes properties of linear functionals rather than linear operators. It will be shown elsewhere that this method of proof yields rather general results in the Banach space context.

Section IV is devoted to a short discussion of certain uniqueness aspects of the problem (1.6). In particular, we will show that if $\operatorname{Re} a(t; x, x)$ is nonnegative, then the uniqueness question is closely related to the question whether or not a certain space of "test functions" Φ is dense in W. This will yield uniqueness in a highly noncoercive example not covered by Theorem 4.9 of [10]. In Section V we will discuss the possibility of obtaining some other existence theorems using techniques analogous to those used in Section III.

Other methods applicable to various kinds of noncoercive and nonmonotone situations are available in literature (among others, [2, 3, 19]) (cf. [20, 22, 23] in the context of variational inequalities). These methods do not seem to apply to our examples. Let us also mention that [1] contains a modification of Lions' projection theorem applicable to certain nonlinear perturbations of linear monotone operators. Some uniqueness theorems with possibly noncoercive applications appear in [3–5, 10, 14, 15].

My thanks are due to Professor Robert Carroll for several helpful comments in an earlier version (in [24]) of Section V of this article.

II

Let $(W, \|\cdot\|)$ be a complex Hilbert space with a normed linear subspace $(\Phi, |||\cdot|||)$ whose antidual is $(\Phi', |||\cdot|||')$. Let $E\colon W \times \Phi \to \mathscr{C}$ be a sesquilinear map continuous in the first variable. Thus, we can define a linear map

$$K\colon \Phi \to K[\Phi] \subset W$$

by

$$((w, K\phi)) = E(w, \phi), \qquad \forall w \in W, \quad \forall \phi \in \Phi, \tag{2.1}$$

where $((\cdot, \cdot))$ is the scalar product in W. We are now ready for the following theorem.

THEOREM 2.1. *Let $L \in (\Phi', |||\cdot|||')$ be arbitrary. Assume that there exists a constant $\beta > 0$ such that*

$$\| K\phi \| \geqslant \beta \, |||\phi||| , \qquad \forall \phi \in \Phi. \tag{2.2}$$

Then there exists a $u_L \in W$ such that

$$E(u_L , \phi) = L(\phi), \qquad \forall \phi \in \Phi \tag{2.3}$$

and

$$\| u_L \| \leqslant (1/\beta) \, |||L|||'.$$

Proof. By (2.2), the map $K^{-1}\colon K[\Phi] \to \Phi$ exists and is continuous from the $\|\cdot\|$—topology to the $|||\cdot|||$—topology, and the operator norm of K^{-1} does not exceed $1/\beta$. Thus, $T = LK^{-1}$ is a continuous, antilinear functional on $K[\Phi]$. By the Hahn–Banach theorem [29], T has a continuous norm-preserving extension to W, so that there exists $u_L \in W$ such that

$$((u_L , w)) = LK^{-1}(w), \qquad \forall w \in K[\Phi], \tag{2.4}$$

and

$$\begin{aligned} \| u_L \| &= \text{norm of } T \\ &\leqslant (1/\beta) \, |||L|||'. \end{aligned}$$

If we put $\phi = K^{-1}w$, then the result (2.3) follows from Eqs. (2.1) and (2.4). This completes the proof of the theorem.

That Lions' projection theorem (cf. [4, 18]) follows easily from Theorem 2.1 is shown in [24, 27] wherein the proof of the following theorem is also available.

THEOREM 2.2. *With K defined by* (2.1), *the solution u_L of Eq.* (2.3) *is unique if and only if $K[\Phi]$ is dense in W.*

Let us point out that the proof of Theorem 2.2 does not make use of inequality (2.2). No solution u_L may exist for some or all $L \in (\Phi', ||| \cdot |||')$. In the proof of Theorem 2.2 only the linearity property of L is utilized—L need not even be continuous.

III

We now start developing our main results which will be stated in the form of Theorems 3.1, 3.2, and 3.3. We assume (1.3) and define W by (1.4) and (1.5). For each $t \in [0, T]$, let $G_r(t)$: $V(t) \to H$ be the restriction of a linear operator $G(t)$: $H \to H$ satisfying the following conditions:

(i) each $G(t)$ is one-to-one, positive, continuous and self-adjoint defined on all of H;
(ii) $V(t) \supset$ range of $G_r(t)$, $\forall t \in [0, T]$, and the family $\{G_r(t) \mid t \in [0, T]\}$ is measurable in the sense explained just below the statement (1.9); (3.1)

and

(iii) either $\{G_r(t) \mid t \in [0, T])$ is a uniformly bounded family of operators, each $G_r(t)$ being bounded in the $\|\cdot\|_t$-topology,

or

$\forall t \in [0, T]$, $S(t)\, G_r(t)$ can be extended to a bounded linear operator $\tilde{G}(t)$: $H \to H$ defined on all of H such that the family $\{\tilde{G}(t) \mid t \in [0, T]\}$ is measurable and uniformly bounded. (3.2)

In addition, we need to make the following assumption:

(iv) the family $\{G(t) \mid t \in [0, T]\}$ is weakly C^1. (3.3)

These assumptions imply (cf. [4, 11, 24]) the existence of a family $\{\dot{G}(t) \mid t \in [0, T]\}$ of linear self-adjoint operators in H defined by

$$(\dot{G}(t)\, x, y)_H = \frac{d}{dt}\,(G(t)\, x, y)_H$$

for every fixed pair of elements $x, y \in H$, and both the families $\{G(t)\}$ and $\{\dot{G}(t)\}$ are uniformly bounded and measurable on $[0, T]$.

Since $G(t)$ is positive and self-adjoint, $G(t)^{1/2}$ is well-defined and has the same properties. Occasionally we will need to assume that

$$\text{the family } \{G(t)^{1/2} \mid t \in [0, T]\} \text{ is weakly } C^1. \tag{3.4}$$

This assumption implies (3.3) as well as the existence of a uniformly bounded, measurable family $\{\dot{G}^{1/2}(t) \mid t \in [0, T]\}$ of linear, self-adjoint operators in H such that

$$(\dot{G}^{1/2}(t)\, x, y)_H = \frac{d}{dt}\,(G(t)^{1/2}\, x, y)_H$$

for every fixed pair of elements $x, y \in H$, and

$$\dot{G}(t) = \dot{G}^{1/2}(t)\, G(t)^{1/2} + G(t)^{1/2}\, \dot{G}^{1/2}(t) \tag{3.5}$$

(cf. [11, 24]) as operators in H. One should be cautioned against confusing $\dot{G}^{1/2}(t)$ with $(\dot{G}(t))^{1/2}$. The existence of $(\dot{G}(t))^{1/2}$ is unknown. Neither $\dot{G}(t)$ nor $\dot{G}^{1/2}(t)$ is assumed to be positive, or even lower semibounded.

We recall the definition (1.7) to introduce the new space of "test functions",

$$\Phi = G(F) = \{G(\cdot)\, v(\cdot) \mid v(\cdot) \in F\} \subset F, \tag{3.6}$$

the last set-inclusion statement being a consequence of the assumptions (3.1) and (3.2).

To save writing in future, let us use the notation,

$$Y_\lambda(G) = 2\mathrm{Re}\left[-\int_0^T (G(t)\, v(t), v'(t))_H\, dt\right] + \lambda \int_0^T (G(t)\, v(t), v(t))_H\, dt \tag{3.7}$$

in which λ is a real constant $\geqslant 0$ and $v \in F$. In the sequel we will many a time refer to one or the other of the following three calculations.

(1) Recall (3.3) and the fact that $G(t)$ is self-adjoint to write

$$\begin{aligned} Y_\lambda(G) &= -\int_0^T \left[\frac{d}{dt}(G(t)\, v(t), v(t))_H - (\dot{G}(t)\, v(t), v(t))_H\right] dt \\ &\quad + \lambda \int_0^T |\, G(t)^{1/2}\, v(t)|_H^2\, dt \\ &= |\, G(0)^{1/2}\, v(0)|_H^2 + \int_0^T (\dot{G}(t)\, v(t), v(t))_H\, dt + \lambda \int_0^T |\, G(t)^{1/2}\, v(t)|_H^2\, dt \end{aligned} \tag{3.8}$$

since $v(T) = 0$.

(2) If (3.4), and therefore (3.5), hold, then we can carry the preceding computation a step further if we note that

$$\int_0^T (\dot{G}(t)\, v(t), v(t))_H \, dt$$
$$= \int_0^T [(\dot{G}^{1/2}(t)\, G(t)^{1/2}\, v(t), v(t))_H + (v(t), \dot{G}^{1/2}(t)\, G(t)^{1/2}\, v(t))_H] \, dt$$

so that

$$\begin{aligned} Y_\lambda(G) = {} & |\, G(0)^{1/2}\, v(0)|_H^2 + 2\mathrm{Re} \int_0^T (\dot{G}^{1/2}(t)\, G(t)^{1/2}\, v(t), v(t))_H \, dt \\ & + \lambda \int_0^T |\, G(t)^{1/2}\, v(t)|_H^2 \, dt. \end{aligned} \tag{3.9}$$

(3) If $\lambda > 0$ and (3.4) holds, then we have another way of computing $Y_\lambda(G)$, namely,

$$\begin{aligned} Y_\lambda(G) = {} & - \int_0^T [(G(t)^{1/2}\, v(t), G(t)^{1/2}\, v'(t))_H + (G(t)^{1/2}\, v'(t), G(t)^{1/2}\, v(t))_H] \, dt \\ & + \lambda \int_0^T |\, G(t)^{1/2}\, v(t)|_H^2 \, dt \\ = {} & - \int_0^T \Big[\frac{d}{dt} (G(t)^{1/2}\, v(t), G(t)^{1/2}\, v(t))_H - (\dot{G}^{1/2}(t)\, v(t), G(t)^{1/2}\, v(t))_H \\ & - (G(t)^{1/2}\, v(t), \dot{G}^{1/2}(t)\, v(t))_H \Big] \, dt + \lambda \int_0^T |\, G(t)^{1/2}\, v(t)|_H^2 \, dt \\ = {} & |\, G(0)^{1/2}\, v(0)|_H^2 + \int_0^T \left| \lambda^{1/2}\, G(t)^{1/2}\, v(t) + \frac{1}{\lambda^{1/2}}\, \dot{G}^{1/2}(t)\, v(t) \right|_H^2 dt \\ & - \frac{1}{\lambda} \int_0^T |\, \dot{G}^{1/2}(t)\, v(t)|_H^2 \, dt, \end{aligned} \tag{3.10}$$

because, again, $G(t)^{1/2}$ is self-adjoint, and $v(T) = 0$.

Our next step is to define $E\colon W \times \Phi \to \mathscr{C}$ as

$$E(u, \phi) = - \int_0^T (u(t), v'(t))_H \, dt + \int_0^T a(t; u(t), v(t)) \, dt + \lambda \int_0^T (u(t), v(t))_H \, dt \tag{3.11}$$

where

$$v(t) = G(t)^{-1}\, \phi(t) \tag{3.12}$$

[see (3.6)] and λ is a nonnegative constant. We immediately have, recalling (1.8),

$$E(u, \phi) = \int_0^T ((u(t), - S(t)^{-2}\, v'(t) + \mathscr{A}(t)^*\, v(t) + \lambda S(t)^{-2}\, v(t)))_t \, dt$$

where $\mathscr{A}(t)^*$ is the adjoint of $\mathscr{A}(t)$ in $V(t)$. This shows that $E(\cdot, \cdot)$ is a sesquilinear form continuous in the first variable. Hence, equation (2.1) yields the following equality in W,

$$K\phi = - S^{-2}v' + \mathscr{A}^*v + \lambda S^{-2}v, \qquad \forall \phi \in \Phi, \tag{3.13}$$

with obvious notations.

This is the starting point for obtaining, by more or less similar techniques, a number of existence results, two of which will be described with examples in this section. A few other existence results will be briefly indicated in Section V. Our technique reduces to putting a suitable norm $|||\cdot|||$ on Φ so that the inequality (2.2) is satisfied. It must be borne in mind, however, that different norms on Φ are likely to necessitate different restrictions on the data f and u_0 in order to preserve the continuity of L on Φ. These restrictions appear as regularity conditions on the data.

So let $\Lambda: \Phi \to W$ be an arbitrary linear operator defined on all of Φ. Taking the W-scalar product of both sides of Eq. (3.13) with $\Lambda\phi$, and recalling (3.6) and (3.12) we obtain,

$$\begin{aligned}((\Lambda\phi, K\phi)) = &- (\Lambda\phi, v')_{L^2(H)} + \int_0^T a(t; (\Lambda\phi)(t), v(t))\, dt \\ &+ \lambda(\Lambda\phi, v)_{L^2(H)} \qquad \forall \phi \in \Phi.\end{aligned} \tag{3.14}$$

In this section we will consider the cases $\Lambda = K$ and $\Lambda = UG^{-1}$, U being a linear operator in W, suitably defined later.

First we consider the case $\Lambda = K$. We may obtain directly from (3.13), that $\forall \phi \in \Phi$,

$$\begin{aligned}\| K\phi \|^2 &= \| - S^{-2}v' + \mathscr{A}^*v \|^2 + \lambda[((\mathscr{A}^*v, S^{-2}v)) + ((S^{-2}v, \mathscr{A}^*v))] \\ &\quad - \lambda[((S^{-2}v', S^{-2}v)) + ((S^{-2}v, S^{-2}v'))] + \lambda^2 \| S^{-2}v \|^2 \\ &= \| - S^{-2}v' + \mathscr{A}^*v \|^2 + 2\lambda \operatorname{Re} \int_0^T a(t; S(t)^{-2}\, v(t), v(t))\, dt \\ &\quad + \lambda Y_\lambda(S^{-2})\end{aligned} \tag{3.15}$$

[see (3.7)]. Now we are ready to state our first theorem.

THEOREM 3.1. *Assume the existence of the operators* $G(t)$ *in* H *satisfying the conditions* (3.1) *and* (3.2). *Assume the validity of either the set of conditions* (A) *or the set of conditions* (B) *described below.*

Conditons (A)

(Ai) $S(\cdot)^{-2}$ is weakly C^1 on $[0, T]$.

(Aii) Constants $\lambda > 0$, $\delta_1 > 0$ exist so as to satisfy

$$\operatorname{Re} \int_0^T a(t; S(t)^{-2} v(t), v(t))\, dt + \tfrac{1}{2} \int_0^T (\dot{S}^{-2}(t)\, v(t), v(t))_H\, dt + \frac{\lambda(1-\delta_1)}{2} \int_0^T | S(t)^{-1} v(t)|_H^2\, dt \geqslant 0, \quad \forall v \in F. \tag{3.16}$$

(Aiii)

$$u_0 \in D(S(0)) = V(0), \qquad f \in W = S^{-1}(L^2(H)). \tag{3.17}$$

Conditions (B)

(Bi) $S(\cdot)^{-1}$ is weakly C^1 on $[0, T]$.

(Bii) Constants $\lambda > 0$, $\gamma > 0$ and $\delta_2 > 0$ ($\delta_2 \geqslant 0$ in the constant domain case) exist so as to satisfy

$$\begin{aligned} \operatorname{Re} \int_0^T & a(t; S(t)^{-2} v(t), v(t))\, dt \\ &\geqslant \frac{1}{2\lambda} \int_0^T | \dot{S}^{-1}(t)\, v(t)|_H^2\, dt + \frac{1}{2}(\gamma - 1)\, | S(0)^{-1} v(0)|_H^2 \\ &+ \frac{\delta_2}{2} \int_0^T | S(t)^{-1} v(t)|_H^2\, dt, \qquad \forall v \in F. \end{aligned} \tag{3.18}$$

(Biii) $u_0 \in D(S(0)) = V(0)$ and $f \in L^2(H)$ satisfying

$$f(t) = \left(\frac{1}{\lambda^{1/2}} \dot{S}^{-1}(t) + \lambda^{1/2} S(t)^{-1}\right) g(t) + S(t)^{-1} h(t) \tag{3.19}$$

a.e. in t for some g, $h \in L^2(H)$.

Then there exists a $u_\lambda \in W$ such that $u = u_\lambda$ satisfies Eq. (1.6).

Proof. We intend to apply Theorem 2.1.

When conditions (A) hold, we define a norm $|||\cdot|||$ on Φ by

$$|||\phi|||^2 = | S(0)^{-1} v(0)|_H^2 + \lambda\delta_1 \int_0^T | S(t)^{-1} v(t)|_H^2\, dt$$

which, together with (3.8) and (3.15), yields

$$\begin{aligned} \| K\phi \|^2 = \| - S^{-2}v' + \mathscr{A}^* v \|^2 &+ 2\lambda \operatorname{Re} \int_0^T a(t; S(t)^{-2} v(t), v(t))\, dt \\ &+ \lambda |||\phi|||^2 + \lambda \int_0^T (\dot{S}^{-2}(t)\, v(t), v(t))\, dt \\ &+ \lambda^2(1-\delta_1) \int_0^T | S(t)^{-1} v(t)|_H^2\, dt \end{aligned}$$

which, in view of (3.16), yields (2.2) with $\beta = \lambda^{1/2}$.

Let us now define $L: \Phi \to C$ by

$$L(\phi) = \int_0^T (f(t), v(t))_H \, dt + (u_0, v(0))_H \tag{3.20}$$

and (3.12). We see that, by virtue of (3.17),

$$L(\phi) = \int_0^T (S(t) f(t), S(t)^{-1} v(t))_H \, dt + (S(0) u_0, S(0)^{-1} v(0))_H$$

which shows that L is continuous on $(\Phi, |||\cdot|||)$.

We now turn our attention to the case when conditions (B) hold. We turn Φ into a normed linear space by defining

$$|||\phi|||^2 = \int_0^T \left| \frac{1}{\lambda^{1/2}} \dot{S}^{-1}(t) v(t) + \lambda^{1/2} S(t)^{-1} v(t) \right|_H^2 dt$$
$$+ \delta_2 \int_0^T | S(t)^{-1} v(t)|_H^2 \, dt + \gamma \, | S(0)^{-1} v(0)|_H^2 , \qquad \forall \phi \in \Phi.$$

Indeed,

$$|||\phi||| = 0 \qquad \text{implies} \qquad \int_0^T | S(t)^{-1} v(t)|_H^2 \, dt = 0$$

whenever $[\delta_2 > 0]$ or $[\delta_2 = 0$ and $\dot{S}^{-1}(t) = 0, \forall t]$. Thus, $|||\phi||| = 0$ implies $v = 0$ which implies $\phi = 0$. This definition of $|||\phi|||$ together with (3.10) and (3.15) yields

$$\| K\phi \|^2 = \| - S^{-2} v' + \mathscr{A}^* v \|^2 + 2\lambda \operatorname{Re} \int_0^T a(t; S(t)^{-2} v(t), v(t)) \, dt$$
$$+ \lambda |||\phi|||^2 + \lambda(1 - \gamma) \, | S(0)^{-1} v(0)|_H^2$$
$$- \int_0^T | \dot{S}^{-1}(t) v(t)|_H^2 \, dt - \lambda\delta_2 \int_0^T | S(t)^{-1} v(t)|_H^2 \, dt$$

which, in view of (3.18), again yields (2.2) with $\beta = \lambda^{1/2}$.

It remains to note that with L defined by (3.20), we obtain from (3.19),

$$L(\phi) = \int_0^T \left(g(t), \frac{1}{\lambda^{1/2}} \dot{S}^{-1}(t) v(t) + \lambda^{1/2} S(t)^{-1} v(t) \right)_H dt$$
$$+ \int_0^T (h(t), S(t)^{-1} v(t))_H \, dt + (S(0) u_0 , S(0)^{-1} \quad v(0))_H .$$

If $\delta_2 = 0$ and $\dot{S}^{-1}(t) = 0$, this yields

$$| L(\phi)| \leqslant \left(| g |_{L^2(H)} + \frac{1}{\lambda^{1/2}} | h |_{L^2(H)} + \frac{1}{\gamma^{1/2}} | S(0) u_0 |_H \right) |||\phi||| ,$$

and if $\delta_2 > 0$, we obtain

$$|L(\phi)| \leqslant \Big(|g|_{L^2(H)} + \frac{1}{(\delta_2)^{1/2}} |h|_{L^2(H)} + \frac{1}{\gamma^{1/2}} |S(0)u_0|_H \Big) |||\phi||| ,$$

showing that in either case $L \in (\Phi', |||\cdot|||')$. This completes the proof of the theorem.

We will now illustrate the use of this theorem by giving a few constant domain examples. In the constant domain case, $V(t) = V$, $\forall t \in [0, T]$, and $((\cdot, \cdot))_t$ is denoted by $((\cdot\, , \cdot))_V$. In all these examples we will take $\delta_1 = 1$, $\gamma = 1$ and $\delta_2 = 0$, so that each of (3.16) and (3.18) reduces to

$$\operatorname{Re} \int_0^T a(t; S(t)^{-2} v(t), v(t))\, dt \geqslant 0, \qquad \forall v \in F, \tag{3.21}$$

and (3.17) and (3.19) become equivalent. It may be possible to choose G in various ways. We will, however, choose $G = S^{-2}$. Then (3.21) reduces to, by virtue of (3.12) and (3.6),

$$\operatorname{Re} \int_0^T a(t; \phi(t), S^2\phi(t))\, dt \geqslant 0, \qquad \forall \phi \in \Phi = S^{-2}(F). \tag{3.22}$$

We collect these results in the form of a theorem.

THEOREM 3.2. *Let H be a separable Hilbert space with a dense subspace V which is also a Hilbert space with continuous inclusion injection: $V \to H$. Let $\forall t \in [0, T]$, $0 < T < \infty$, $a(t; \cdot, \cdot)$; $V \times V \to \mathscr{C}$ be a continuous sesquilinear form with the map $t \mapsto a(t; x, y)$; $[0, T] \to \mathscr{C}$ measurable and bounded for every fixed pair of elements $x, y \in V$. Let S: $V \to H$ be Carroll's standard operator. Then, under the assumption* (3.22), *equation* (1.6) *possesses a solution $u \in W = S^{-1}(L^2(H))$ for every pair of data (f, u_0) where $f \in W$ and $u_0 \in V$.*

Theorem 3.2 is an improvement upon Theorem 3.1 (with $n = 2$, $\beta = 1$) of [11] wherein, in place of (3.22), one had to satisfy the more restrictive condition,

$$\operatorname{Re} a(t; \phi(t), S^2\phi(t)) \geqslant \alpha \|\phi(t)\|_V^2, \qquad \forall \phi \in S^{-2}(F) \tag{3.23}$$

for almost all $t \in [0, T]$, for some positive constant α independent of t. Clearly (3.22) is satisfied whenever (3.23) is (thus, examples 3.4 and 3.6 of [11] serve to illustrate noncoercive applications of our Theorem 3.2 also). That the converse is not true is shown by the following noncoercive examples.

EXAMPLE 3.1. Our first example enlarges upon example 3.4 of [11], which is associated with the Schrödinger-type differential equation $u_t + gu_x + hu = f$. Thus, let $H = L^2((0, 1); \mathscr{C})$, $V = H_0^1$, and let $\forall u, v \in V$,

$$a(t; u, v) = \int_0^1 g(t, \xi)\, u_x(\xi)\, \bar{v}(\xi)\, d\xi + \int_0^1 h(t, \xi)\, u(\xi)\, \bar{v}(\xi)\, d\xi, \tag{3.24}$$

where g and h are real-valued functions on $[0, T] \times [0, 1]$ satisfying

$$\begin{aligned} &2h - h_{xx} - g_x \geqslant 0, \qquad 2h + g_x \geqslant 0 \qquad \forall (t, x) \in [0, T] \times [0, 1] \\ &\text{and} \\ &g(t, 1) \leqslant 0, \qquad g(t, 0) \geqslant 0 \qquad \forall t \in [0, T]. \end{aligned} \tag{3.25}$$

We assume that h, h_x, h_{xx}, g, g_x are all continuous though, in fact, less may be needed. Explicit examples of such g and h are easily available, e.g.,

(1) $g(t, x) = t^2x - t^2x^2$, $h(t, x) = -t^2x + (3t^2/2)$, in which case

$$2h - g_x = 2t^2 = 2h - h_{xx} - g_x, \qquad 2h + g_x = 4t^2(1 - x),$$

or

(2) $g(t, x) = (t^2/2) - t^2x^2$, $h(t, x) = t^2x$, in which case

$$2h - g_x = 4t^2x = 2h - h_{xx} - g_x, \qquad 2h + g_x = 0,$$

or

$$\text{(3)} \quad g(t, x) = (e^{3x} - e^3x)\, \theta(t), \qquad h(t, x) = (\tfrac{1}{2}\, e^3 - e^{3x})\, \theta(t), \tag{3.26}$$

$\theta(t)$ being a bounded, measurable, nonnegative, real-valued function on $[0, T]$. In this case,

$$\left.\begin{aligned} 2h - h_{xx} - g_x &= (2e^3 + 4e^{3x})\, \theta(t), \\ 2h + g_x &= e^{3x}\theta(t), \\ 2h - g_x &= (2e^3 - 5e^{3x})\, \theta(t). \end{aligned}\right\} \tag{3.27}$$

Note that, if $\theta(t) \neq 0$, then $2h - g_x$ is actually negative in a neighborhood of $x = 1$.

As shown in [11], (3.24) leads to

$$\begin{aligned} 2\mathrm{Re}\, a(t; \phi(t), S^2\phi(t)) = \int_0^1 (2h - h_{xx} - g_x)\, |\, \phi\, |^2\, dx + \int_0^1 (2h + g_x)\, |\, \phi_x\, |^2\, dx \\ - g(t, 1)\, |\, \phi_x(t, 1)|^2 + g(t, 0)\, |\, \phi_x(t, 0)|^2 \end{aligned} \tag{3.28}$$

which is nonnegative by virtue of the conditions (3.25). Thus, (3.22) is satisfied.

We want to emphasize that (3.24) defines a noncoercive form. Indeed, as shown in [11],

$$2\mathrm{Re}\, a(t; u, u) = \int_0^1 (2h - g_x) \mid u \mid^2 dx, \qquad \forall u \in V,$$

which can actually be made positive as well as negative by choosing u suitably in the particular case when g and h are given by (3.26) (see comments following (3.27)). This is a highly noncoercive situation. Also, with some of the specific g's and h's that we have provided above, $2h - h_{xx} - g_x$ and $2h + g_x$ are not bounded away from zero in $[0, T] \times [0, 1]$. Thus, as (3.28) shows, no strictly positive constant α is available to satisfy (3.23).

EXAMPLE 3.2. Let Ω be an open connected region of $\mathscr{R}^p$, p being a positive integer and $\mathscr{R}$ denoting the system of real numbers. An element of Ω will be denoted by $\mathbf{x}$. Let $H = L^2(\Omega; \mathscr{C})$, and let the Hilbert space V be a dense subset of H with continuous inclusion injection : $V \to H$. Let a map $g : \Omega \to \mathscr{C}$ satisfy the conditions,

$$\mathrm{Re}\, g \geqslant 0,$$
$$v \in V \qquad \text{implies} \qquad gv \in V.$$

With $\theta(t)$ a nonnegative, integrable, real-valued function on $[0, T]$, define

$$a(t; u, v) = \theta(t)\, ((u, gv))_V \qquad \forall u, v \in V, \quad \forall t \in [0, T]. \tag{3.29}$$

Then,

$$\begin{aligned} &\forall \phi \in \Phi = S^{-2}(F), \\ a(t; \phi(t), (S^2\phi)(t)) &= \theta(t)\, ((\phi(t), g(S^2\phi)(t)))_V \\ &= \theta(t)\, (S^2\phi(t), gS^2\phi(t))_H \\ &= \theta(t) \int_\Omega \overline{g(\mathbf{x})} \mid S^2\phi(t, \mathbf{x})|^2\, d\mathbf{x}. \end{aligned}$$

Hence, we obtain (3.22). Since g is not given to be bounded away from zero, we can easily construct a noncoercive example. For example, let

$$\Omega = (0, 1) \times (0, 1) \subset \mathscr{R}^2.$$

Let $V = H_0^1$, $g(x, y) = x$ and $\theta(t) = 1$. Then $\forall u$, $v \in V$, (3.29) yields,

$$a(t; u, v) = \int_0^1\int_0^1 x[u\bar{v} + u_x\bar{v}_x + u_y\bar{v}_y]\, dx\, dy + \int_0^1\int_0^1 u_x\bar{v}\, dx\, dy$$

so that

$$\mathrm{Re}\, a(t; u, u) = \int_0^1\int_0^1 x[\mid u \mid^2 + \mid u_x \mid^2 + \mid u_y \mid^2]\, dx\, dy.$$

This shows that if $\alpha > 1$, then

$$\operatorname{Re} a(t; u, u) < \alpha \| u \|_V^2, \qquad \forall u \in V,$$

and if $0 < \alpha \leqslant 1$, then

$$\operatorname{Re} a(t; u, u) < \alpha \| u \|_V^2, \qquad \forall u \in C_0^\infty((0, \alpha/2) \times (0, \alpha/2); \mathscr{C}).$$

Thus, $a(t; \cdot, \cdot)$ is noncoercive.

Let us recall that we have so long been dealing with the case $\Lambda = K$ in (3.14), and our results until now depended on the equality (3.15). We will now examine some of the consequences of letting $\Lambda = UG^{-1}$ in (3.14) where $U\colon W \to W$ is defined by a family $\{U(t) \mid t \in [0, T]\}$ of uniformly bounded, continuous operators $U(t)\colon V(t) \to V(t)$ by the relation $(Uw)(t) = U(t)\, w(t)$ $\forall w \in W$ for almost all $t \in [0, T]$. We assume that $\forall w \in W$, the map

$$t \mapsto \| U(t)\, w(t) \|_t \colon [0, T] \to \mathscr{C}$$

is measurable, and that each $U(t)$ is the restriction to $V(t)$ of a positive, self-adjoint, one-to-one operator $\tilde{U}(t)\colon H \to H$ with domain H. We further assume that $\tilde{U}(\cdot)$ is weakly C^1 on $[0, T]$.

In view of (3.12) we have $\forall \phi \in \Phi$

$$\Lambda\phi = UG^{-1}\phi = Uv. \tag{3.30}$$

We then have, from (3.14), that $\forall \phi \in \Phi$,

$$((Uv, K\phi)) = -(Uv, v')_{L^2(H)} + \int_0^T a(t; U(t)\, v(t), v(t))\, dt + \lambda(Uv, v)_{L^2(H)},$$

and, therefore, using the notation (3.7),

$$\begin{aligned} 2\operatorname{Re}((Uv, K\phi)) &= 2\operatorname{Re} \int_0^T a(t; U(t)\, v(t), v(t))\, dt + Y_{2\lambda}(\tilde{U}) \\ &\leqslant 2 \| Uv \| \, \| K\phi \| . \end{aligned} \tag{3.31}$$

THEOREM 3.3. *Assume the existence of operators $G(t)$ in H satisfying* (3.1) *and* (3.2). *Then, under each of the following two sets of conditions* (C) *and* (D), *there exists a $u_\lambda \in W$ such that $u = u_\lambda$ satisfies Eq.* (1.6) *for a given $f \in L^2(H)$ and a given $u_0 \in H$ such that $\tilde{U}^{-1} f$ and $\tilde{U}(0)^{-1/2} u_0$ are defined.*

Conditions (C)

(Ci) $\tilde{U}(\cdot)$ is weakly C^1 on $[0, T]$.

(Cii) Constants $\lambda \geqslant 0$ and $\alpha > 0$ exist such that

$$\operatorname{Re} \int_0^T a(t; U(t)\, v(t), v(t))\, dt + \tfrac{1}{2} \int_0^T (\dot{\tilde{U}}(t)\, v(t), v(t))_H\, dt + \lambda \int_0^T |\, \tilde{U}(t)^{1/2}\, v(t)|_H^2\, dt$$
$$\geqslant \alpha \int_0^T \|\, U(t)\, v(t)\|_t^2\, dt, \qquad \forall v \in F. \tag{3.32}$$

Conditions (D)

(Di) $\tilde{U}(\cdot)^{1/2}$ is weakly C^1 on $[0, T]$.

(Dii) Constants $\lambda > 0$, $\alpha > 0$ exist such that

$$\begin{aligned} &\operatorname{Re} \int_0^T a(t; U(t)\, v(t), v(t))\, dt - \frac{1}{4\lambda} \int_0^T |\, \dot{\tilde{U}}^{1/2}(t)\, v(t)|_H^2\, dt \\ &\qquad \geqslant \alpha \int_0^T \|\, U(t)\, v(t)\|_t^2\, dt, \qquad \forall v \in F. \end{aligned} \tag{3.33}$$

This completes the statement of the theorem.

Proof. Define the norm $|||\cdot|||$ on $\Phi = G(F)$ by

$$|||\, \phi\, |||^2 = \|\, Uv\, \|^2 + |\, \tilde{U}(0)^{1/2}\, v(0)|_H^2 \tag{3.34}$$

[(recall (3.12)]. Substituting the value of $Y_{2\lambda}(\tilde{U})$ given by (3.8) in (3.31), and utilizing (3.32), we obtain,

$$\begin{aligned} \|\, Uv\, \|\, \|\, K\phi\, \| &\geqslant \tfrac{1}{2}\, |\, \tilde{U}(0)^{1/2}\, v(0)|_H^2 + \alpha \int_0^T \|\, U(t)\, v(t)\|_t^2\, dt \\ &\geqslant \min(\tfrac{1}{2}, \alpha)\, |||\, \phi\, |||^2 \qquad \text{by (3.34).} \end{aligned} \tag{3.35}$$

We arrive at the same result (3.35), if conditions (D) hold, by substituting in (3.31) the value of $Y_{2\lambda}(\tilde{U})$ given by (3.10), and then utilizing (3.33). By (3.34), $|||\, \phi\, ||| \geqslant \|\, Uv\, \|$, and, therefore, (3.35) yields (2.2) with $\beta = \min(\frac{1}{2}, \alpha)$. It remains to note that, under our hypotheses, (3.20) yields,

$$|\, L(\phi)| \leqslant \left| \int_0^T ((\tilde{U}^{-1} f)\,(t), (Uv)\,(t))_H\, dt \right| + |(\tilde{U}(0)^{-1/2}\, u_0\, , \tilde{U}(0)^{1/2}\, v(0))_H\, |$$

so that L is continuous on Φ [see (3.34)]. The proof of the theorem is now completed by applying Theorem 2.1.

In Theorem 3.3, $U(t)$ may be variable even when $V(t) = V$ (a constant domain) for all $t \in [0, T]$. An example of the other extreme, namely when $\tilde{U}(t)$

does not vary but $V(t)$ does, occurs when $\tilde{U}(t)$ is the identity operator for all t. In this latter situation, either of the inequalities (3.32) and (3.33) reduces to

$$\operatorname{Re} \int_0^T a(t; v(t), v(t))\, dt \geqslant \alpha \| v \|^2, \qquad v \in F,$$

which is satisfied if the forms $a(t; \cdot, \cdot)$ constitute a coercive family. We have thus recovered the familiar existence theorem (cf. [4]) under coercivity in the variable domain situation.

EXAMPLE 3.3. Here we give another version of Example 3.2. This would be a weaker version except for the different regularity conditions imposed on f and u_0. So, we let Ω be an open region of $\mathscr{R}^p$, p being a positive integer. Let $H = L^2(\Omega; \mathscr{C})$ and V a dense subspace of H, V itself being a Hilbert space with the inclusion injection: $V \to H$ continuous. Let $g: \Omega \to \mathscr{R}$ be a bounded, strictly positive (except on a set of measure zero), measurable function such that $v \in V$ implies $gv \in V$. Define $a(t; \cdot, \cdot)$ by (3.29), where $\theta: [0, T] \to \mathscr{C}$ is bounded and measurable with $\operatorname{Re} \theta(t) \geqslant \alpha > 0$ for almost all $t \in [0, T]$, for some constant α. Defining $U: W \to W$ by

$$(Uw)(t, \mathbf{x}) = g(\mathbf{x})\, w(t, \mathbf{x}) \qquad \forall (t, \mathbf{x}) \in [0, T] \times \Omega,$$

we obtain

$$a(t; (Uv)(t), v(t)) = \theta(t) \, \|(Uv)(t)\|_V^2$$

so that (3.32) and (3.33) are both satisfied.

IV

We know (cf. [4, 18]) that in the constant domain situation, (weak) uniqueness of solution of Eq. (1.6) is obtained if we have, $\forall t \in [0, T]$,

$$\operatorname{Re} a(t; v, v) > 0 \qquad \text{for all nonzero } v \in V. \tag{4.1}$$

An improved result is given in [10] according to which, in the variable domain situation, (weak) uniqueness is obtained if

$$\begin{gathered} \operatorname{Re} a(t; v, v) \geqslant 0 \qquad \forall v \in V(t) \quad \text{for almost all } t \in [0, T], \\ \text{and} \\ \{S(t)^{-2} \mid t \in [0, T]\} \qquad \text{is weakly } C^1. \end{gathered} \tag{4.2}$$

Our discussion in this section will bring out a close relationship between the uniqueness question and the denseness of F or Φ in W. A significant by-product of our discussion will be to obtain uniqueness in a constant domain example which does not satisfy (4.2). We begin by noticing that (2.1), (3.7), (3.11), and (3.12) yield

$$\begin{aligned}\operatorname{Re}((\phi, K\phi)) &= \operatorname{Re} E(\phi,\phi)\\ &= \operatorname{Re}\int_0^T a(t; G(t)\, v(t), v(t))\, dt + \tfrac{1}{2} Y_{2\lambda}(G), \qquad \forall \phi \in \Phi. \end{aligned} \tag{4.3}$$

LEMMA 4.1. *Suppose at least one of the two following conditions hold.*

(1) *For all nonzero $v \in F$* (see (1.7))

$$\operatorname{Re}\int_0^T a(t; G(t)\, v(t), v(t))\, dt + \tfrac{1}{2}\int_0^T (\dot{G}(t)\, v(t), v(t))_H\, dt > 0. \tag{4.4}$$

(2) *A positive constant λ exists such that*

$$\operatorname{Re}\int_0^T a(t; G(t)\, v(t), v(t))\, dt > \frac{1}{4\lambda}\int_0^T |\, \dot{G}^{1/2}(t)\, v(t)|_H^2\, dt \tag{4.5}$$

for all nonzero $v \in F$.

Then, as subsets of W, each of Φ and $K[\Phi]$ is dense in the other.

Proof. Deny the validity of the lemma. Then there exists a nonzero $\phi \in \Phi$ such that $((\phi, K\phi)) = 0$. Then, in the respective cases when $G(\cdot)$ or $G(\cdot)^{1/2}$ is weakly C^1 on $[0, T]$, (4.3) and (3.8) contradict (4.4), whereas (4.3) and (3.10) contradict (4.5). Thus, the lemma is true.

THEOREM 4.1. *Suppose* (4.4) *or* (4.5) *holds. Then a solution of Eq.* (1.6) *is unique if and only if Φ is dense in W.*

Proof. If Φ is dense in W, then by Lemma 4.1, $K[\Phi]$ is dense in W. Now define $E(\cdot, \cdot)$ by (3.11), $L(\cdot)$ by (3.20) and (3.12), and then apply the "if part" of Theorem 2.2 to obtain the desired uniqueness.

If the solution of (1.6) is unique, then by the "only if part" of Theorem 2.2, $K[\Phi]$ is dense in W. Lemma 4.1 now says that Φ is dense in W.

This completes the proof of the theorem.

COROLLARY 4.1. *If*

$$\operatorname{Re}\int_0^T a(t; v(t), v(t))\, dt > 0, \qquad \forall \textit{ nonzero } v \in F, \tag{4.6}$$

then the solution of (1.6) *is unique if and only if F is dense in W.*

Proof. Let $G(t)$ be the identity operator for each t. Then $\Phi = F$, and (4.6) implies each of (4.4) and (4.5). The corollary now follows from Theorem 4.1.

We are now able to recover the following known result.

COROLLARY 4.2. *Let* $H = L^2((0, 1); \mathscr{C})$. *Suppose* $V(t) = V$, $\forall t \in [0, T]$, *and assume* (4.1). *Then the solutions of Eq.* (1.6) *are unique.*

Proof. If $v \in F$, $v \neq 0$, then $v(t) \neq 0$ on a subset of $[0, T]$ of positive measure. Thus, (4.1) implies (4.6). All that remains to show is that F is dense in $W = L^2([0, T]; V)$. This is true because, if $X = C_0^\infty((0, T); \mathscr{C}) \otimes V$, then, on one hand $X \subset F \subset W$, and, on the other hand, X is dense in $L^2((0, T); \mathscr{C}) \otimes V$ which (cf. [29]), in turn, is dense in $L^2((0, T); V)$. This proves the corollary.

A converse to Corollary 4.2 is possible. As proved in [4] and [18], the condition (4.1) makes the solution of equation (1.6) unique in the constant domain case. Since (4.1) implies (4.6), Corollary 4.1 tells us that F must be dense in $W = L^2([0, T]; V)$. If $V(t)$ is not constant, then one may in general expect to encounter some limitation on the way $V(t)$ varies if F is to continue to remain dense in $W = L^2([0, T]; V(t)) = S^{-1}(L^2(H))$. We will obtain in Corollary 4.3 a set of sufficient conditions under which F remains dense in W. But first we quote the following result from [10].

THEOREM 4.2. *Under the conditions* (4.2), *solutions of equation* (1.6) *are* (*weakly*) *unique.*

Proof. This theorem is merely a particular case of Theorem 4.9 of [10].

COROLLARY 4.3. *If* $S(\cdot)^{-2}$ *is weakly* C^1 *on* $[0, T]$, *then* F *is dense in* W.

Proof. Construct the variable domain problem—with the given H and $V(t)$—which consists of solving Eq. (1.6) with $a(t; \cdot, \cdot)$ defined as

$$a(t; u, v) = ((u, v))_t \qquad \forall u, v \in V(t) \quad \forall t \in [0, T].$$

This is a coercive problem satisfying (4.2), and so its solution is unique by Theorem 4.2. Since (4.6) is satisfied, Corollary 4.1 yields the desired result.

Inasmuch as the converse of Corollary 4.3 is not known to be true, Corollary 4.1 may be looked upon as an improvement on Theorem 4.2 above or on Theorem 4.9 of [10].

A constant domain specialization of Theorem 4.1 is obtained if, in (4.4) or (4.5), we put $G(t) = G = S^{-2}$, $\forall t \in [0, T]$, with

$$V(t) = V \qquad \forall t \in [0, T].$$

THEOREM 4.3. *Suppose, in the constant domain case,*

$$\operatorname{Re} \int_0^T a(t; \phi(t), S^2\phi(t))\, dt > 0 \qquad \textit{for all nonzero } \phi \in \Phi = S^{-2}(F). \tag{4.7}$$

Then a solution of Eq. (1.6) *is unique if and only if* Φ *is dense in*

$$W = L^2([0, T]; V).$$

We now proceed to give an example illustrating how Theorem 4.3 may be applied. Let $H = L^2((0, 1); \mathscr{C})$ and suppose $C_0^\infty((0, 1); \mathscr{C}) \subset V \subset H$. Since $C_0^\infty((0, T); \mathscr{C}) \otimes C_0^\infty((0, 1); \mathscr{C})$ is dense in $L^2((0, T) \times (0, 1); \mathscr{C})$, it follows that W is dense in $L^2([0, T]; H)$. Therefore, by properties of S^{-1}, $S^{-1}(W)$ is dense in $W = S^{-1}(L^2(H))$ in the topology of W. Since S^{-1}: $W \to W$ is continuous in the topology of W, and since F is dense in W, we have $S^{-1}(F)$ dense in $S^{-1}(W)$, and therefore in W. Repetition of the same argument leads to the result that $\Phi = S^{-2}(F)$ is dense in W. Thus, the stage is set to apply Theorem 4.3 to the following example.

EXAMPLE 4.1. This is Example 3.1 with $H = L^2((0, 1); \mathscr{C})$, $V = H_0^1$, $a(t; \cdot, \cdot)$ given by (3.24) and (3.26) with $\theta(t) > 0$ $\forall t$. As seen in (3.27), $2h - h_{xx} - g_x > 0$ and $2h + g_x > 0$ $\forall (t, x) \in [0, T] \times [0, 1]$. Thus, (3.28) yields (4.7). Since Φ is dense in W, Theorem 4.3 shows that the solutions of the corresponding Eq. (1.6) are unique. Let us emphasize, as before, that this example is highly noncoercive, because Re $a(t; u, u)$ takes up positive as well as negative values. In particular, Theorem 4.9 of [10] does not apply to this example.

V

Equation (3.15), together with Eqs. (3.8) and (3.10), have played a significant role in obtaining Theorem 3.1. These equations can obviously be exploited in various other ways to obtain other results of which we describe only a few in this section. How useful they are remains to be investigated.

LEMMA 5.1 (cf. [24]). *Under the assumption* (4.6),

$$\| - S^{-2}v' + \mathscr{A}^*v \| = 0 \quad \textit{implies} \quad v = 0, \qquad \textit{if} \qquad v \in F.$$

Proof. If $v \in F$ satisfies $\| - S^{-2}v' + \mathscr{A}^*v \| = 0$, then

$$S(t)^{-2}\, v'(t) = \mathscr{A}(t)^*\, v(t)$$

for almost all $t \in [0, T]$, and $\mathscr{A}(\cdot)^* v(\cdot) \in D(S^2)$ = domain of S^2. We, therefore, have

$$((\mathscr{A}(t)^* v(t), v(t)))_t = ((S(t)^{-2} v'(t), v(t)))_t \qquad \text{a.e. in } t$$

and this yields

$$\int_0^T \overline{a(t; v(t), v(t))}\, dt = \int_0^T (v'(t), v(t))_H\, dt,$$

the bar denoting complex conjugate. We thus have

$$\begin{aligned} \operatorname{Re} \int_0^T a(t; v(t), v(t))\, dt &= \tfrac{1}{2} \int_0^T (v, v)'_H (t)\, dt \\ &= -\tfrac{1}{2} \mid v(0)|_H^2 \qquad \text{since } v(T) = 0 \\ &\leqslant 0. \end{aligned}$$

Hence, by (4.6), $v = 0$, proving the lemma.

If (4.6) is true, then Lemma 5.1 allows us to define the norm $|||\cdot|||$ on Φ by

$$|||\phi|||^2 = \| -S^{-2}v' + \mathscr{A}^* v \|^2 + \Delta, \tag{5.1}$$

where $v = S^2\phi$, and Δ is a suitable nonnegative quantity. To obtain the inequality (2.2) we may now adopt one of the following two courses.

(1) If constants $\delta_1 \geqslant 0$, $\delta_2 \geqslant 0$ and $\lambda \geqslant 0$ exist such that

$$\begin{aligned} &2\lambda \operatorname{Re} \int_0^T a(t; S(t)^{-2} v(t), v(t))\, dt \\ &\quad + \lambda(1 - \delta_1) \mid S(0)^{-1} v(0)|_H^2 + \lambda \int_0^T (\dot{S}^{-2}(t)\, v(t), v(t))_H\, dt \\ &\quad + \lambda^2(1 - \delta_2) \int_0^T \mid S(t)^{-1} v(t)|_H^2\, dt \geqslant 0, \qquad \forall v \in F, \end{aligned} \tag{5.2}$$

then we choose

$$\Delta = \lambda\delta_1 \mid S(0)^{-1} v(0)|_H^2 + \lambda^2\delta_2 \int_0^T \mid S(t)^{-1} v(t)|_H^2\, dt.$$

Then (3.8), (3.15), and (5.1) will imply (2.2).

(2) If constants $\lambda > 0$, $\gamma_1 \geqslant 0$, $\gamma_2 \geqslant 0$, $\gamma_3 \geqslant 0$ exist such that

$$\begin{aligned} &2\lambda \operatorname{Re} \int_0^T a(t; S(t)^{-2} v(t), v(t))\, dt \\ &\quad + \lambda(1 - \gamma_1) \mid S(0)^{-1} v(0)|_H^2 \\ &\quad + \lambda(1 - \gamma_2) \int_0^T \left| \lambda^{1/2} S(t)^{-1} v(t) + \frac{1}{\lambda^{1/2}} \dot{S}^{-1}(t)\, v(t) \right|_H^2 dt \\ &\quad - (1 + \gamma_3) \int_0^T \mid \dot{S}^{-1}(t)\, v(t)|_H^2\, dt \geqslant 0, \qquad \forall v \in F, \end{aligned} \tag{5.3}$$

then we choose

$$\Delta = \lambda\gamma_1 \mid S(0)^{-1} v(0)|_H^2 + \lambda\gamma_2 \int_0^T \left| \lambda^{1/2} S(t)^{-1} v(t) + \frac{1}{\lambda^{1/2}} \dot{S}^{-1}(t) v(t) \right|_H^2 dt$$
$$+ \gamma_3 \int_0^T \mid \dot{S}^{-1}(t) v(t)|_H^2 dt.$$

Then (3.10), (3.15), and (5.1) will imply (2.2).

We can thus use Theorem 2.1 again to yield existence of solution of Eq. (1.6) under the hypothesis (5.2) or (5.3). Indeed, different choices of combinations of the constants δ_1, δ_2, γ_1, γ_2 and γ_3 in (5.2) or (5.3) would amount to different kinds of hypotheses, yielding different theorems. In each case, of course, an appropriate set of regularity conditions on f and u_0 has to be worked out so that the antilinear form L on Φ, defined by (3.20), is continuous in the topology given by (5.1). One choice of regularity conditions which is consistent with all choices of $\Delta \geqslant 0$ in (5.1) is to have $f \in L^2(H)$, $u_0 \in H$ satisfying

$$f = g' + S^2 \mathscr{A} g, \qquad g(0) = u_0 \tag{5.4}$$

for some $g \in W$ with

$$g' \in L^2(H) \qquad \text{and} \qquad \mathscr{A}(\cdot)\, g(\cdot) \in D(S(\cdot)^2) = \text{domain of } S^2, \tag{5.5}$$

because then, keeping (3.20) in view, we have $\forall \phi \in \Phi$,

$$\begin{aligned} L(\phi) &= \int_0^T (g'(t), v(t))_H\, dt + \int_0^T (S(t)^2 \mathscr{A}(t)\, g(t), v(t))_H\, dt + (g(0), v(0))_H \\ &= \int_0^T (g, v)'_H (t)\, dt - \int_0^T (g(t), v'(t))_H\, dt \\ &\quad + \int_0^T ((\mathscr{A}(t)\, g(t), v(t)))_t\, dt + (g(0), v(0))_H \\ &= ((g, - S^{-2} v' + \mathscr{A}^* v)) \qquad \text{since } v(T) = 0. \end{aligned} \tag{5.6}$$

We have thus arrived at the following proposition.

PROPOSITION 5.1. *Assume that* $S(\cdot)^{-1}$ *is weakly* C^1 *on* $[0, T]$, *and assume that the conditions* (4.6), (5.4), *and* (5.5) *are satisfied. Then there exists a solution of equation* (1.6) *under each of the following two hypotheses.*

(1) $$\operatorname{Re} \int_0^T a(t; S(t)^{-2} v(t), v(t))\, dt \geqslant 0, \qquad \forall v \in F$$

and (5.7)

$$\operatorname{Re} \int_0^T (\dot{S}^{-1}(t)\, S(t)^{-1} v(t), v(t))_H\, dt \geqslant 0, \qquad \forall v \in F;$$

(2) *a constant $\lambda > 0$ exists such that $\forall v \in F$*

$$\operatorname{Re} \int_0^T a(t; S(t)^{-2} v(t), v(t))\, dt \geqslant \frac{1}{2\lambda} \int_0^T | \dot{S}^{-1}(t)\, v(t)|_H^2\, dt. \tag{5.8}$$

Proof. Define $\Phi = S^{-2}(F)$ and let the norm on Φ be given by (5.1). (5.7) implies that (5.2) is true with $\delta_1 = 0 = \delta_2$ [see the lines preceding (3.9)]. (5.8) implies that (5.3) is true with $\gamma_1 = \gamma_2 = \gamma_3 = 0$. In both cases $\Delta = 0$ and, as noted above, inequality (2.2) results. Once we note (5.6), the proof is completed by applying Theorem 2.1.

In this proposition, assumption (4.6) was made in addition to the hypotheses (5.7) and (5.8). No such additional assumption is needed in the following more general version.

PROPOSITION 5.2. *Assume* (3.1), (3.2), (3.4), *and one of the following two conditions:*

$$\left.\begin{aligned}
&\text{(1) Real constants } \delta_1 \text{ and } \delta_2 \text{ exist such that}\\
&\qquad \operatorname{Re}(\dot{G}^{1/2}(t)\, G^{-1/2}(t)\, v, v)_H \geqslant \delta_1 \,| v |_H^2, \qquad \forall v \in D(G(t)^{-1/2})\\
&\text{(a similar hypothesis arises in [12, 13, 28]; (also cf. [11, 24]), and}\\
&2\operatorname{Re} \int_0^T a(t; G(t)\, v(t), v(t))\, dt\\
&\quad > \delta_2 \int_0^T | G(t)^{1/2}\, v(t)|_H^2\, dt - | G(0)\, v(0)|_H^2 \quad \text{for all nonzero } v \in F;
\end{aligned}\right\} \tag{5.9}$$

or

$$\left.\begin{aligned}
&\text{(2) A positive constant } \delta > 0 \text{ exists such that}\\
&2\operatorname{Re} \int_0^T a(t; G(t)\, v(t), v(t))\, dt\\
&\quad > \delta \int_0^T | \dot{G}^{1/2}(t)\, v(t)|_H^2\, dt - | G(0)^{1/2}\, v(0)|_H^2 \quad \text{for all nonzero } v \in F.
\end{aligned}\right\} \tag{5.10}$$

Assume that $f \in L^2(H)$ and $u_0 \in H$ are given satisfying the regularity conditions

$$f = g' + cg + S^2 \mathscr{A} g, \qquad u_0 = g(0) \tag{5.11}$$

where g satisfies (5.5) *and c is a constant satisfying*

$$c \geqslant \begin{cases} -\delta_1 - \frac{1}{2}\delta_2 & \text{if (5.9) is true}\\ \dfrac{1}{2\delta} & \text{if (5.10) is true.} \end{cases} \tag{5.12}$$

Then Eq. (1.6) *has a solution.*

Proof. Let the norm on $\Phi = G(F)$ be given by

$$|||\phi||| = \| - S^{-2}v' + \mathscr{A}^*v + cS^{-2}v \| \qquad \forall\phi \in \Phi \tag{5.13}$$

where v is given by (3.12). Indeed, $|||\phi||| = 0$ implies

$$(S(t)^2 \mathscr{A}(t)^* v(t), G(t) v(t))_H = (v'(t), G(t) v(t))_H - c(v(t), G(t) v(t))_H$$

for almost all t in $[0, T]$. This, in turn, implies

$$2\mathrm{Re} \int_0^T a(t; G(t) v(t), v(t))\, dt = -Y_{2c}(G)$$

according to (3.7). Equations (3.9) and (3.10) now respectively yield, together with (5.12),

$$\begin{aligned} 2\mathrm{Re} &\int_0^T a(t; G(t) v(t), v(t))\, dt \\ &+ | G(0)^{1/2} v(0)|_H^2 + 2\mathrm{Re} \int_0^T (\dot{G}^{1/2}(t) G(t)^{1/2} v(t), v(t))_H\, dt \\ &- (2\delta_1 + \delta_2) \int_0^T | G(t)^{1/2} v(t)|_H^2\, dt \leqslant 0 \end{aligned}$$

and

$$\begin{aligned} 2\mathrm{Re} &\int_0^T a(t; G(t) v(t), v(t))\, dt \\ &+ | G(0)^{1/2} v(0)|_H^2 + \int_0^T |(2c)^{1/2} G(t)^{1/2} v(t) + 1/(2c)^{1/2} \dot{G}^{1/2}(t) v(t)|_H^2\, dt \\ &- \delta \int_0^T | \dot{G}^{1/2}(t) v(t)|_H^2\, dt \leqslant 0, \end{aligned}$$

which contradict (5.9) and (5.10), respectively, unless $v = 0$. Hence, $|||\phi||| = 0$ implies $\phi = 0$.

Now let $E(\cdot, \cdot)$ be defined by (3.11) with λ replaced by c. Then (3.13) and (5.13) imply $\| K\phi \| = |||\phi|||$ so that Theorem 2.1 can be applied to obtain existence of solution of Eq. (1.6) for $\lambda = c$ and, therefore, for all λ (by standard arguments of change of variable). We can deduce the continuity of L on Φ from (5.11) and (5.5) by calculations similar to those which led to (5.6). This establishes Proposition 5.2.

We gain very little from Proposition 5.2 so long as we have to contend with the conditions (5.11). We have, however, made our point that various existence results can be deduced by means of the technique presented in this paper. As yet another instance of this technique, we could replace the zeros on the right sides of the inequalities (5.2) and (5.3) by $\alpha \| v \|^2$, α being a positive constant. (5.1) then has to be replaced by

$$||| \phi |||^2 = \| -S^{-2}v' + \mathscr{A}^* v \|^2 + \Delta + \alpha \| v \|^2, \tag{5.14}$$

so as to ensure that inequality (2.2) is obtained. (5.14) automatically produces the result, "$||| \phi ||| = 0$ implies $\phi = 0$". So this procedure has the advantage of not requiring any "extra" condition such as (4.6) needed in Lemma 5.1 and Proposition 5.1. The regularity conditions (5.4) and (5.5) suffice, and are possibly more than necessary to prove continuity of L on Φ, as is clear from (5.6). In fact, (5.6) and (5.14) yield $| L(\phi)| \leqslant \| g \| \, ||| \phi |||$. Whether, in this way, we end up with a better result than Proposition 5.1 remains to be investigated. Another as yet unknown factor is the possibility of arranging in a well-defined order all the various existence results obtainable by our technique.

References

1. H. Brézis, Perturbations non-linéaires d'opérateurs maximaux monotones, *C. R. Acad. Sci. Paris* Serie A, **269** (1969), 566–569.
2. H. Brézis, On some degenerate nonlinear parabolic equations, appearing in Nonlinear Functional Analysis, *in* "Proc. Symp. in Pure Math.," Vol. XVIII, Part I, 28–38, Amer. Math. Soc., 1970.
3. F. E. Browder, Nonlinear elliptic boundary value problems and the generalized topological degree, *Bull. Amer. Math. Soc.* **76** (1970), 999–1005.
4. R. Carroll, "Abstract Methods in Partial Differential Equations," Harper and Row, New York, 1969.
5. R. Carroll, On the nature of weak solutions and some abstract Cauchy problems, *Bull. Amer. Math. Soc.* **72** (1966), 1068–1072.
6. R. Carroll, On the propagator equation, *Illinois J. Math.* **11** (1967), 506–527.
7. R. Carroll, Some variable domain problems in abstract evolution equations, Proc. Conf. Eqs. of Evolution and Nonlinear Semigroups, pp. 11–24, Univ. of of Kentucky, 1969.
8. R. Carroll, On the structure of some abstract differential problems I and II, *Annali di Mat.* **72** (1966), 305–318; **81** (1969), 93–110.
9. R. Carroll and J. Cooper, Solutions des équations d'évolutions abstraites à domaine variable, *C. R. Acad. Sci., Paris* **270** (1970), 319–321.
10. R. Carroll and J. Cooper, Remarks on some variable domain problems in abstract evolution equations, *Math. Ann.* **188** (1970), 143–164.
11. R. Carroll and T. Mazumdar, Solutions of some possibly noncoercive evolution problems with regular data, *J. Appl. Anal.* **1** (1972), 381–395.
12. R. Carroll and E. State, Solutions faibles des équations d'évolution hyperboliques à domaine variable, *C. R. Acad. Sci. Paris* **271** (1970), 931–933.
13. R. Carroll and E. State, Existence theorems for some weak abstract variable domain hyperbolic problems, *Can. J. Math.* **23** (1971), 611–626.
14. J. Cooper, Thesis, University of Illinois, Urbana, Illinois, 1967.

15. J. COOPER, Evolution equations in Banach space with variable domain, *J. Math. Anal. Appl.* **36** (1971), 151–171.
16. J. COOPER, Un problème relatif aux équations d'évolution abstraites, *C. R. Acad. Sci., Paris* **268** (1967), 422–424.
17. J. COOPER, Two-point problems for abstract evolution equations, *J. Differential Equations* **9** (1971), 453–495.
18. J. L. LIONS, "Équations différentielles-opérationelles," Springer–Verlag, Berlin, 1961 (Grundlehren der mathematischen Wissenschaften, 111).
19. J. L. LIONS, "Quelques méthodes de résolution des problèmes aux limites non linéaires," Dunod and Gauthier–Villars, Paris, 1969.
20. J. L. LIONS, "Optimal control of systems governed by partial differential equations," translated into English by S. K. Mitter, Springer–Verlag, Berlin, 1971.
21. J. L. LIONS AND E. MAGENES, "Problèmes aux limites non homogènes et applications," Vol. 1, Dunod, Paris, 1968.
22. J. L. LIONS AND G. STAMPACCHIA, Inéquations variationelles non coercives, *C. R. Acad. Sci., Paris* **261** (1965), 25–27.
23. J. L. LIONS AND G. STAMPACCHIA, Variational inequalities, *Comm. Pure Appl. Math.* **20** (1967), 493–519.
24. T. MAZUMDAR, Thesis, University of Illinois at Urbana–Champaign, 1971.
25. T. MAZUMDAR, Existence results for some possibly noncoercive evolution problems with regular data, *Not. Amer. Math. Soc.* **18** (1971), 164.
26. T. MAZUMDAR, A generalized projection theorem, *Not. Amer. Math. Soc.* **19** (1972), A-15.
27. T. MAZUMDAR, Generalized projection theorem with application to linear noncoercive equations and some nonlinear situations, *J. Math. Anal. Appl.* **43** (1973), 72–100.
28. E. STATE, Thesis, University of Illinois at Urbana–Champaign, 1971.
29. F. TREVES, "Topological Vector Spaces, Distributions and Kernels," Academic Press, New York, 1967.